Vector operations so far

Scalar multiplication:

c **v** = "vector parallel to **v** with length scaled by a factor of *c*"

Vector Addition:

a + **b** = "if **a** and **b** are drawn tail to head, then **a** + **b** is the vector that goes from the tail of **a** to the head of **b**" (resultant/combined force)

12.3 Dot Products

If **a** = $\langle a_1, a_2, a_3 \rangle$ and **b** = $\langle b_1, b_2, b_3 \rangle$

Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note: The dot product gives a number (scalar).

Entry Task: **a** = < 3, 1, 2 >, c = 4

b = - **i** + 6**j** + 5**k**

Compute

1. c**a**
2. unit vector in the direction of **a.**
3. **a** + **b**
4. **a** · **b**

$\boxed{1}$ $4\langle 3,1,2\rangle = \boxed{\langle 12,4,8\rangle}$

$\boxed{2}$ $|\vec{a}| = \sqrt{3^2+(1)^2+(2)^2} = \sqrt{14}$

$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{14}}\langle 3,1,2\rangle = \boxed{\left\langle \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right\rangle}$

$\boxed{3}$ $\langle 3,1,2\rangle + \langle -1,6,5\rangle = \boxed{\langle 2,7,7\rangle}$

$\boxed{4}$ $\langle 3,1,2\rangle \cdot \langle -1,6,5\rangle$

$= -3+6+10 = \boxed{13}$

Basic fact list:

- Manipulation facts (works like regular multiplication):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b})$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$$

$2(\langle 3,1,2\rangle \cdot \langle -1,6,5\rangle) = 2 \cdot 13 = 26$

$\langle 6,2,4\rangle \cdot \langle -1,6,5\rangle = -6 + 12 + 20 = 26$

$\uparrow$

ONLY MULTIPLY ONE, DOES NOT DISTRIBUTE

$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$= \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$

- Helpful fact:

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$$

$\langle 3,1,2\rangle \cdot \langle 3,1,2\rangle$

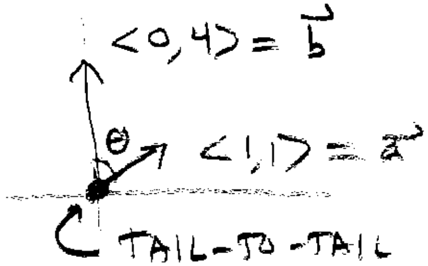
$(3)^2 + (1)^2 + (2)^2 = 13 = |\vec{a}|^2$

The most important fact:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta),$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \le \theta \le \pi)$

Ex



$\vec{a}\cdot\vec{b} = 0+4 = 4$

$|\vec{a}| = \sqrt{1+1} = \sqrt{2}$

$|\vec{b}| = \sqrt{0+16} = 4$

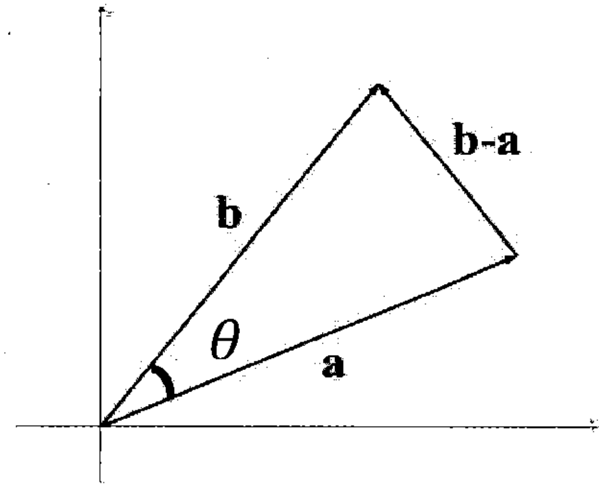
$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \quad 4 = \sqrt{2}\cdot 4\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} \overset{\text{SAME}}{\longleftrightarrow} \frac{\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\pi/4 \text{ radians}}$

$45°$



Proof (not required):

By the Law of Cosines:

$$|\mathbf{b}-\mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

The left-hand side expands to

$$\begin{aligned} |\mathbf{b}-\mathbf{a}|^2 &= (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) \\ &= \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} \\ &= |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2 \end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides gives

$$-2\mathbf{a}\cdot\mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Divide by -2 to get the result. (QED)

Most important consequence:
If **a** and **b** are orthogonal, then

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Also: If **a** and **b** are parallel, then
$\mathbf{a} \cdot \mathbf{b} = |a||b|$ or $\mathbf{a} \cdot \mathbf{b} = -|a||b|$.

Ex) $\vec{a} = \langle 1, 1, 2 \rangle$, $\vec{b} = \langle 3, 2, -5 \rangle$, $\vec{c} = \langle -6, 4, 1 \rangle$, $\vec{d} = \langle 9, 6, -15 \rangle$ — ARE ANY PARALLEL? ARE ANY ORTHOGONAL?

$\vec{b}$ and $\vec{d}$ are parallel $\vec{d} = 3\vec{b}$

$\vec{a} \cdot \vec{b} = 3 + 2 - 10 = -5$, NOT ORTHOGONAL

$\vec{a} \cdot \vec{c} = -6 + 4 + 2 = 0$, YES, ORTHOGONAL

$\vec{b} \cdot \vec{c} = -18 + 8 - 5 = -15$, NOT ORTHOGONAL

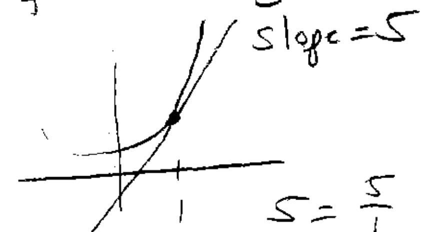
Example: Find a vector that is orthogonal to the tangent line to $y = x^3 e^{(2x-2)}$ at $x = 1$.

$$y' = x^3 e^{(2x-2)} \cdot 2 + 3x^2 e^{2x-2}$$

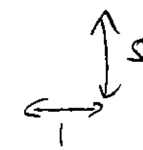
$$y' = 2x^3 e^{(2x-2)} + 3x^2 e^{2x-2}$$

$$y'(1) = 2(1)^3 e^0 + 3(1)^2 e^0$$

$$= 5 = \text{slope of tangent}$$



$5 = \frac{5}{1}$



VECTOR PARALLEL TO TANGENT $= \boxed{\langle 1, 5 \rangle}$

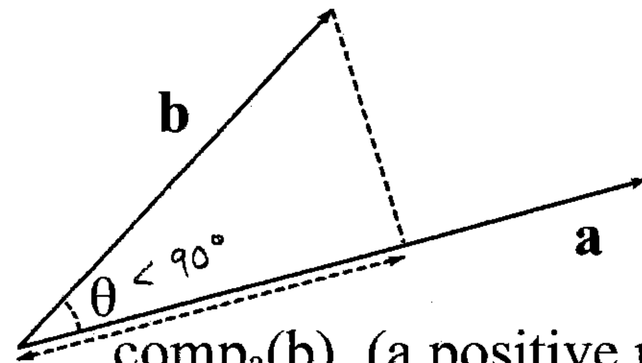
"NEGATIVE RECIPROCAL"

VECTOR ORTHOGONAL TO TANGENT $= \boxed{\langle -5, 1 \rangle}$

IS THE DOT PRODUCT ZERO??? $\boxed{\text{YES!}}$

Projections:

GIVEN $\vec{a}$ AND $\vec{b}$ DRAWN TAIL-TO-TAIL, WHAT IS THIS LENGTH?



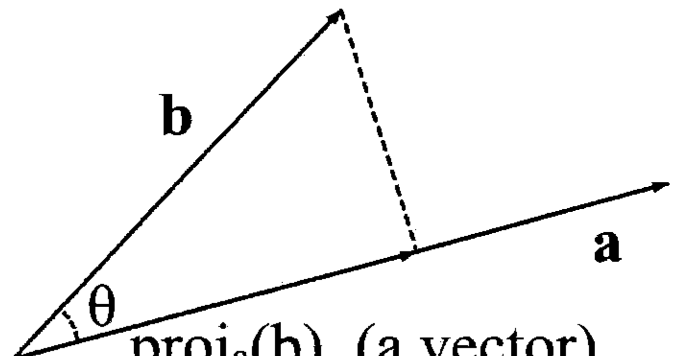
compa(b) (a positive number)

component projection of $\vec{b}$ onto $\vec{a}$

NOTE

$$\cos\theta = \frac{\text{comp}_{\vec{a}}(\vec{b})}{|\vec{b}|} \qquad \vec{a}\cdot\vec{b} = |\vec{a}|\underbrace{|\vec{b}|\cos\theta}$$

$$\Rightarrow \boxed{\text{comp}_{\vec{a}}(\vec{b}) = |\vec{b}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

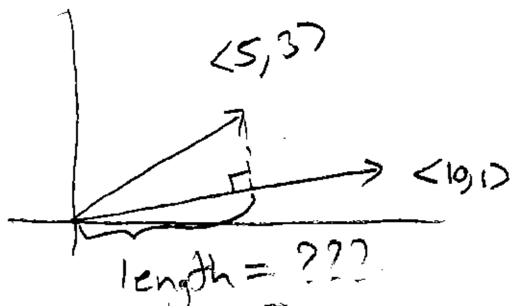


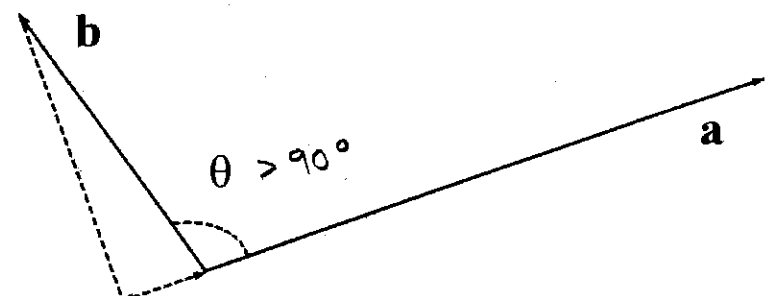
proja(b) (a vector)

vector projection of $\vec{b}$ onto $\vec{a}$

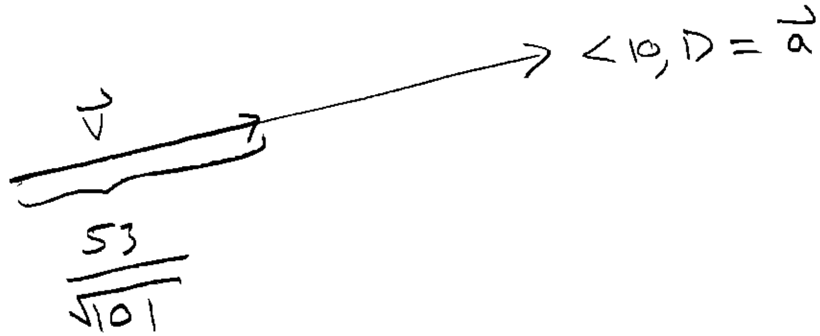
EXAMPLE $\quad \vec{a} = \langle 10, 1\rangle \quad \vec{b} = \langle 5, 3\rangle$

$$\text{comp}_{\langle 10,1\rangle}(\langle 5,3\rangle) = \frac{50+3}{\sqrt{10^2+1^2}} = \boxed{\frac{53}{\sqrt{101}}}$$





compa(b) (a negative number)



FIND $\vec{v} = ???$ $\left(\text{proj}_{\vec{a}}(\vec{b})\right)$

RESCALE!

$$\vec{v} = \underbrace{\frac{53}{\sqrt{101}}}_{\substack{\text{SCALE}\\ \text{TO}\\ \text{THIS}\\ \text{LENGTH}}}\Bigg(\underbrace{\frac{1}{|\vec{a}|}\vec{a}}_{\substack{\text{UNIT}\\ \text{VECTOR}\\ \text{IN}\\ \text{DIRECTION}\\ \text{OF}\\ \vec{a}}}\Bigg) = \frac{53}{\sqrt{101}}\left(\frac{1}{\sqrt{101}}\langle 10, 1\rangle\right) = \frac{53}{101}\langle 10, 1\rangle = \left\langle \frac{530}{101}, \frac{53}{101}\right\rangle$$

SUMMARY

$$\underbrace{\text{comp}_{\vec{a}}(\vec{b})}_{} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\left(\frac{1}{|\vec{a}|}\vec{a}\right)$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

$$= \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$$

12.4 The Cross Product

We define the <u>cross product</u>, or <u>vector product</u>, for two 3-dimensional vectors,
$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and
$\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$,
by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Ex: $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$((2)(2) - (3)(0))\boldsymbol{i} - ((1)(2) - (-1)(0))\boldsymbol{j} + ((1)(3) - (-1)(2))\boldsymbol{k}$$

$$(4 - 0)\vec{i} - (2 - 0)\vec{j} + (3 + 2)\vec{k}$$

$$= \boxed{\langle 4, -2, 5 \rangle} = \vec{a} \times \vec{b}$$

$$\langle 4, -2, 5 \rangle \cdot \langle 1, 2, 0 \rangle$$
$$= 4 - 4 + 0 = 0 \leftarrow \star$$

$$\langle 4, -2, 5 \rangle \cdot \langle -1, 3, 2 \rangle$$
$$= -4 - 6 + 10 = 0 \leftarrow \star$$

You do: $\boldsymbol{a} = \langle 1,3,-1 \rangle$, $\boldsymbol{b} = \langle 2,1,5 \rangle$.

Compute $\boldsymbol{a} \times \boldsymbol{b}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{vmatrix}$$

$$= (15 - -1)\vec{i} - (5 - -2)\vec{j} + (1 - 6)\vec{k}$$

$$= \langle 16, -7, -5 \rangle$$

$$\langle 16,-7,-5 \rangle \cdot \langle 1,3,-1 \rangle = 16-21+5 = 0 \leftarrow \star$$

$$\langle 16,-7,-5 \rangle \cdot \langle 2,1,5 \rangle = 32-7-25 = 0 \leftarrow \star$$

Most important fact:

The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* **a** and **b**.

"proof":

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

ALWAYS!!

$$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1} = 0$$

SIMILARLY FOR $\langle b_1, b_2, b_3 \rangle$!

Right-hand rule

If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

$\boxed{\text{NOTE}}$ $\langle 1,3,1\rangle \times \langle 2,1,5\rangle = \langle 16,-7,-5\rangle$ ← EARLIER EXAMPLE

$\langle 2,1,5\rangle \times \langle 1,3,1\rangle = \langle -16,7,5\rangle$

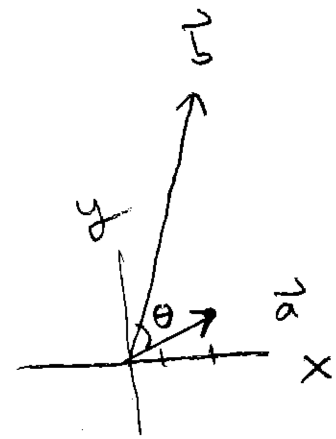
ORDER MATTERS

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

Ex)

$$\left.\begin{aligned} \vec{a} &= \langle 2,1,0\rangle \\ \vec{b} &= \langle 3,10,0\rangle \end{aligned}\right\}$$

ABOVE VIEW



DOES $\vec{a} \times \vec{b}$ POINT UPWARD OR DOWNWARD? ← UPWARD

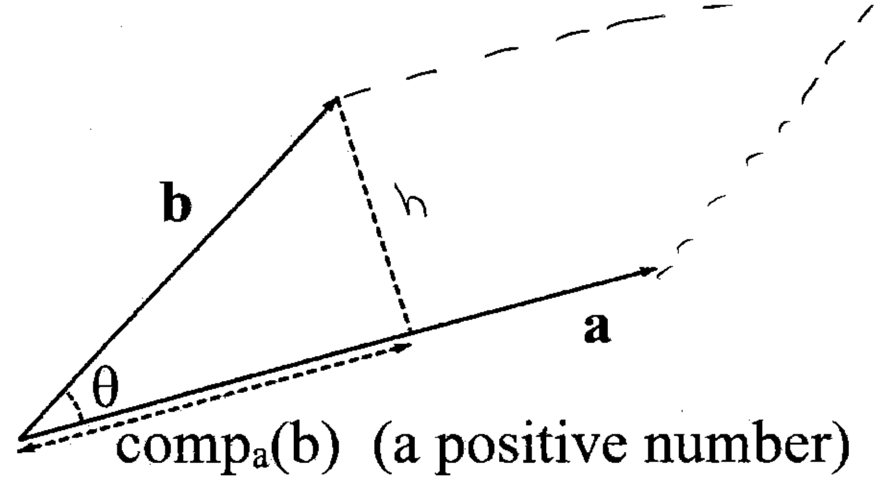
DOES $\vec{b} \times \vec{a}$ POINT UPWARD OR DOWNWARD? ← DOWNWARD

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$:

Through some algebra and using the dot product rule, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \le \theta \le \pi)$



LONG PROOF

$$|\vec{a} \times \vec{b}|^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$$
$$= \text{BIG ALGEBRA MESS}$$
$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$
$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$
$$= |\vec{a}|^2 |\vec{b}|^2 (\underbrace{1 - \cos^2\theta}_{\sin^2\theta})$$
$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$$

Note: $|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$ is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\sin\theta = \frac{h}{|\vec{b}|} \Rightarrow h = |\vec{b}| \sin\theta$$

$$\boxed{\begin{aligned} \text{AREA OF PARALLELOGRAM} &= |\vec{a}| \cdot h \\ &= |\vec{a}| |\vec{b}| \sin\theta \\ &= |\vec{a} \times \vec{b}|. \end{aligned}}$$

AREA OF TRIANGLE FORMED BY $\vec{a}$ AND $\vec{b}$ IS $\frac{1}{2} |\vec{a} \times \vec{b}|$